

A Class of Isotropic Mean Berwald Metrics

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Abstract

In this paper, we find a condition on (α, β) -metrics under which the notions of isotropic S-curvature, weakly isotropic S-curvature and isotropic mean Berwald curvature are equivalent.¹

Keywords: (α, β) -metric, isotropic S-curvature, isotropic E-curvature.

1 Introduction

The S-curvature is introduced by Shen for a comparison theorem on Finsler manifolds [10]. Recent study shows that the S-curvature plays a very important role in Finsler geometry [11][13]. A Finsler metric F is said to have isotropic S-curvature if $\mathbf{S} = (n+1)cF$, where $c = c(x)$ is a scalar function on an n -dimensional manifold M .

Taking twice vertical covariant derivatives of the S-curvature gives rise the E -curvature. A Finsler metric F with vanishing E -curvature called weakly Berwald metric. In [1], Bácsó-Yoshikawa study some weakly Berwald metrics. Also, F is called to have isotropic E -curvature if $\mathbf{E} = \frac{n+1}{2}cF^{-1}\mathbf{h}$, for some scalar function c on M , where \mathbf{h} is the angular metric. It is easy to see that every Finsler metric of isotropic S-curvature is an isotropic E -curvature. Now, is the equation $\mathbf{S} = (n+1)cF$ equivalent to the equation $\mathbf{E} = \frac{n+1}{2}cF^{-1}\mathbf{h}$?

Recently, Cheng-Shen prove that a Randers metric $F = \alpha + \beta$ is of isotropic S-curvature if and only if it is of isotropic E -curvature [3]. Then, Chun-Huan-Cheng extend this equivalency to the Finsler metric $F = \alpha^{-m}(\alpha + \beta)^{m+1}$ for every real constant m , including Randers metric [2]. In [8], Lee-Lee prove that this notions are equivalent for the Finsler metrics in the form $F = \alpha + \alpha^{-1}\beta^2$.

All of above metrics are special Finsler metrics so-called (α, β) -metrics. An (α, β) -metric is a scalar function on TM defined by $F := \alpha\phi(s)$, $s = \beta/\alpha$ where $\phi = \phi(s)$ is a C^∞ on $(-b_0, b_0)$ with certain regularity, α is a Riemannian metric and β is a 1-form on a manifold M [12]. A natural question arises:

Is being of isotropic S-curvature is equivalent to being of isotropic E-curvature for (α, β) -metrics?

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In [7], Deng-Wang find the formula of the S -curvature of homogeneous (α, β) -metrics. Then Cheng-Shen classify (α, β) -metrics of isotropic S -curvature [4].

Let $F = \alpha\phi(s)$ be an (α, β) -metric on a manifold M of dimension n , where $s = \frac{\beta}{\alpha}$, $\alpha = \sqrt{a_{ij}y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on M . For an (α, β) -metric, put

$$\begin{aligned} Q &= \frac{\phi'}{\phi - s\phi'}, \\ \Delta &= 1 + sQ + (b^2 - s^2)Q', \\ \Phi &= -(Q - sQ')\{n\Delta + 1 + sQ\} - (b^2 - s^2)(1 + sQ)Q'', \end{aligned}$$

Using the same method as in [4], we give an affirmative answer to the above question for almost all (α, β) -metrics. More precisely, we prove the following.

Theorem 1.1 *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an (α, β) -metric on a manifold M . Let us define*

$$\Xi := \frac{(b^2 Q + s)\Phi}{\Delta^2}. \quad (1)$$

Suppose that Ξ is not constant. Then F is of isotropic S -curvature if and only if it is of isotropic E -curvature.

It is remarkable that if $\Xi = 0$, then F reduces to a Riemannian metric. But, in general, it is still an open problem if Theorem 1.1 is true when Ξ is a constant.

Example 1.2 *The above mentioned (α, β) -metric correspond to $\phi = 1 + s$, $\phi = (1 + s)^{m+1}$ and $\phi = 1 + s^2$, respectively. Using a Maple program shows that for all these metrics Ξ is not constant.*

2 Preliminaries

Let $F = F(x, y)$ be a Finsler metric on an n -dimensional manifold M . There is a notion of distortion $\tau = \tau(x, y)$ on TM associated with a volume form $dV = \sigma(x)dx$, which is defined by

$$\tau(x, y) = \ln \frac{\sqrt{\det(g_{ij}(x, y))}}{\sigma(x)}.$$

Then the S -curvature is defined by

$$\mathbf{S}(x, y) = \frac{d}{dt} \left[\tau(c(t), \dot{c}(t)) \right]_{t=0},$$

where $c(t)$ is the geodesic with $c(0) = x$ and $\dot{c}(0) = y$ [6]. From the definition, we see that the S -curvature $\mathbf{S}(y)$ measures the rate of change in the distortion on $(T_x M, F_x)$ in the direction $y \in T_x M$.

Let $\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}$ denote the spray of F and $dV_{BH} = \sigma(x)dx$ be the Busemann-Hausdorff volume form on M , where the spray coefficients G^i are defined by

$$G^i(y) := \frac{1}{4}g^{il}(y)\left\{\frac{\partial^2[F^2]}{\partial x^k \partial y^l}(y)y^k - \frac{\partial[F^2]}{\partial x^l}(y)\right\}, \quad y \in T_x M.$$

Then the S-curvature is given by

$$\mathbf{S} = \frac{\partial G^m}{\partial y^m} - y^m \frac{\partial}{\partial x^m}(\ln \sigma).$$

The E-curvature $\mathbf{E} = E_{ij}dx^i \otimes dx^j$ is given by

$$E_{ij} = \frac{1}{2} \frac{\partial^2 S}{\partial y^i \partial y^j}.$$

Definition 2.1 Let (M, F) be a n -dimensional Finsler manifold. Then

- (a) F is of isotropic S-curvature if $\mathbf{S} = (n+1)cF$,
- (b) F is of weak isotropic S-curvature if $\mathbf{S} = (n+1)cF + \eta$,
- (c) F is of isotropic E-curvature if $\mathbf{E} = \frac{n+1}{2}cF^{-1}\mathbf{h}$,

where $c = c(x)$ is a scalar function on M , $\eta = \eta_i(x)y^i$ is a 1-form on M and \mathbf{h} is the angular metric [9].

Consider the (α, β) -metric $F = \alpha\phi\left(\frac{\beta}{\alpha}\right)$ where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on a manifold M . In [4], the following is proved:

Lemma 2.2 Let (M, F) be a n -dimensional Finsler manifold. Suppose that $F = \alpha\phi(\beta/\alpha)$ be an (α, β) -metric. Then

$$\frac{\partial G^m}{\partial y^m} = y^m \frac{\partial}{\partial x^m}(\ln \sigma_\alpha) + 2\Psi(r_0 + s_0) - \alpha^{-1} \frac{\Phi}{2\Delta^2}(r_{00} - 2\alpha Q s_0), \quad (2)$$

where

$$\begin{aligned} Q &= \frac{\phi'}{\phi - s\phi'}, \quad \Delta = 1 + sQ + (b^2 - s^2)Q', \quad \Psi = \frac{Q'}{2\Delta} \\ \Phi &= -(Q - sQ')\{n\Delta + 1 + sQ\} - (b^2 - s^2)(1 + sQ)Q''. \end{aligned}$$

For an (α, β) -metric, put

$$\begin{aligned} r_{ij} &:= \frac{1}{2}(b_{i|j} + b_{j|i}), \quad s_{ij} := \frac{1}{2}(b_{i|j} - b_{j|i}), \\ r_j &:= b^i r_{ij}, \quad s_j := b^i s_{ij}, \quad r_{i0} := r_{ij}y^j, \quad s_{i0} := s_{ij}y^j, \quad r_0 := r_j y^j, \quad s_0 := s_j y^j. \end{aligned}$$

Let \bar{G}^i denote the spray coefficients of α . We have the following formula for the spray coefficients G^i of F [6]:

$$G^i = \bar{G}^i + \alpha Q s^i_0 + \Theta \left\{ -2Q\alpha s_0 + r_{00} \right\} \frac{y^i}{\alpha} + \Psi \left\{ -2Q\alpha s_0 + r_{00} \right\} b^i,$$

where $s^i_j := a^{ih} s_{hj}$, $s^i_0 := s^i_j y^j$ and $r_{00} := r_{ij} y^i y^j$. In [4], Cheng-Shen find the S-curvature as follows

$$\mathbf{S} = \left\{ 2\Psi - \frac{f'(b)}{bf(b)} \right\} (r_0 + s_0) - \alpha^{-1} \frac{\Phi}{2\Delta^2} (r_{00} - 2\alpha Q s_0). \quad (3)$$

Recently, Cheng-Shen characterize (α, β) -metrics with isotropic S-curvature and proved the following.

Lemma 2.3 ([4]) *Let $F = \alpha\phi(\beta/\alpha)$ be an (α, β) -metric on an n -manifold. The, F is of isotropic S-curvature $\mathbf{S} = (n+1)cF$, if and only if one of the following holds*

(i) β satisfies

$$r_{ij} = \varepsilon \left\{ b^2 a_{ij} - b_i b_j \right\}, \quad s_j = 0, \quad (4)$$

where $\varepsilon = \varepsilon(x)$ is a scalar function, and $\phi = \phi(s)$ satisfies

$$\Phi = -2(n+1)k \frac{\phi \Delta^2}{b^2 - s^2}, \quad (5)$$

where k is a constant. In this case, $c = k\varepsilon$.

(ii) β satisfies

$$r_{ij} = 0, \quad s_j = 0. \quad (6)$$

In this case, $c = 0$.

It is remarkable that, Cheng-Wang-Wang prove that the condition $\Phi = 0$ characterizes the Riemannian metrics among (α, β) -metrics [5]. Hence, in the continue, we suppose that $\Phi \neq 0$.

3 Proof of Theorem 1.1

First, we find the formula of E-curvature of (α, β) -metrics. After a long and tedious computations, we obtain the following.

Proposition 3.1 *Let $F = \alpha\phi(\frac{\beta}{\alpha})$ be an (α, β) -metric. Put $\Omega := \frac{\Phi}{2\Delta^2}$. Then the E-curvature of F is given by the following*

$$\begin{aligned} E_{ij} = & C_1 b_i b_j + C_2 (b_i y_j + b_j y_i) + C_3 y_i y_j + C_4 a_{ij} + C_5 (r_{i0} b_j + r_{j0} b_i) \\ & + C_6 (r_{i0} y_j + r_{j0} y_i) + C_7 r_{ij} + C_8 (s_i b_j + s_j b_i) + C_9 (s_i y_j + s_j y_i) \\ & + C_{10} (r_i b_j + r_j b_i) + C_{11} (r_i y_j + r_j y_i), \end{aligned} \quad (7)$$

where

$$\begin{aligned}
C_1 &:= \frac{1}{2\alpha^3\Delta^2} \left\{ \Phi\alpha Q''s_0 + 2\alpha\Delta^2\Psi''r_0 - \Delta^2\Omega''r_0 + 2\Delta^2\alpha\Omega''Qs_0 \right. \\
&\quad \left. + 4\Delta^2\alpha\Omega'Q's_0 + 2\alpha\Delta^2\Psi''s_0 \right\}, \\
C_2 &:= \frac{-1}{2\alpha^4\Delta^2} \left\{ 2\alpha\Delta^2\Psi''s_0 - 2\Omega'\Delta^2r_0 + 2\Omega'\Delta^2\alpha Qs_0 - \Delta^2\Omega''sr_0 \right. \\
&\quad + 2\Delta^2\alpha\Omega''sQs_0 + 4\Delta^2\alpha\Omega'Q's_0s + 2\alpha\Delta^2\Psi'r_0 + 2\alpha\Delta^2\Psi''sr_0 \\
&\quad \left. + 2\alpha\Delta^2\Psi''ss_0 + \Phi\alpha Q's_0 + \Phi\alpha Q''s_0s \right\}, \\
C_3 &:= \frac{1}{4\alpha^5\Delta^2} \left\{ 4\Delta^2s^2\Omega''\alpha Qs_0 - 2\Delta^2s^2\Omega''r_0 + 12\alpha\Delta^2\Psi'sr_0 + 12\alpha\Delta^2\Psi'ss_0 \right. \\
&\quad + 4\alpha\Delta^2\Psi''s^2r_0 + 4\alpha\Delta^2\Psi''s^2s_0 + 8\Delta^2s^2\Omega'\alpha Q's_0 + 2\Phi\alpha Q''s_0s^2 \\
&\quad \left. - 10\Omega'\Delta^2sr_0 + 12\Omega'\Delta^2s\alpha Qs_0 + 6\Phi\alpha Q's_0s - 3\Phi r_0 \right\}, \\
C_4 &:= \frac{-1}{4\alpha^3\Delta^2} \left\{ 4\alpha\Delta^2\Psi'ss_0 - \Phi r_0 - 2\Omega'\Delta^2sr_0 + 4\Omega'\Delta^2s\alpha Qs_0 \right. \\
&\quad \left. + 4\alpha\Delta^2\Psi'sr_0 + 2\Phi\alpha Q's_0s \right\}, \\
C_5 &:= \frac{-\Omega'}{\alpha^2}, \quad C_6 := \frac{2\Delta^2s\Omega' + \Phi}{2\alpha^3\Delta^2}, \quad C_7 := \frac{-\Phi}{2\alpha\Delta^2}, \\
C_8 &:= \frac{1}{2\alpha\Delta^2} \{ 2\Omega'\Delta^2Q + 2\Delta^2\Psi' + \Phi Q' \}, \\
C_9 &:= \frac{-s}{\alpha}C_8, \quad C_{10} := \frac{\Psi'}{\alpha}, \quad C_{11} := \frac{-s}{\alpha}C_{10}.
\end{aligned}$$

The formula of E -curvature of Randers metrics and Kropina metrics computed from Proposition 3.1 coincides with the one computed in [1].

It is easy to see that F is of isotropic mean Berwald curvature if and only if F is of weak isotropic S-curvature. Hence, we consider an (α, β) -metric $F = \alpha\phi(\beta/\alpha)$ with weak isotropic S-curvature, $\mathbf{S} = (n+1)cF + \eta$, where $\eta = \eta_i(x)y^i$ is a 1-form on underlying manifold M . Using the same method used in [4], one can obtains that the condition weak isotropic S-curvature $\mathbf{S} = (n+1)cF + \eta$ is equivalent to the following equation

$$\alpha^{-1} \frac{\Phi}{2\Delta^2} (r_{00} - 2\alpha Qs_0) - 2\Psi(r_0 + s_0) = -(n+1)cF + \tilde{\theta}, \quad (8)$$

where

$$\tilde{\theta} := -\frac{f'(b)}{bf(b)}(r_0 + s_0) - \eta. \quad (9)$$

To simplify the equation (8), we choose special coordinates $\psi : (s, u^A) \rightarrow (y^i)$ as follows

$$y^1 = \frac{s}{\sqrt{b^2 - s^2}}\bar{\alpha}, \quad y^A = u^A, \quad (10)$$

where

$$\bar{\alpha} = \sqrt{\sum_{A=2}^n (u^A)^2}.$$

Then

$$\alpha = \frac{b}{\sqrt{b^2 - s^2}} \bar{\alpha}, \quad \beta = \frac{bs}{\sqrt{b^2 - s^2}} \bar{\alpha}. \quad (11)$$

Fix an arbitrary point x . Take a local coordinate system at x as in (10). We have

$$\begin{aligned} r_1 &= br_{11}, & r_A &= br_{1A}, \\ s_1 &= 0, & s_A &= bs_{1A}. \end{aligned}$$

Let

$$\begin{aligned} \bar{r}_{10} &:= \sum_{A=2}^n r_{1A} y^A, & \bar{s}_{10} &:= \sum_{A=2}^n s_{1A} y^A, & \bar{r}_{00} &:= \sum_{A,B=2}^n r_{AB} y^A y^B, \\ \bar{r}_0 &:= \sum_{A=2}^n r_A y^A, & \bar{s}_0 &:= \sum_{A=2}^n s_A y^A. \end{aligned}$$

Put

$$\tilde{\theta} = t_i y^i - \eta_i y^i.$$

Then t_i are given by

$$t_1 = -\frac{f'(b)}{f(b)} r_{11}, \quad t_A = -\frac{f'(b)}{f(b)} (r_{1A} + s_{1A}). \quad (12)$$

From (10), we have

$$r_0 = \frac{sbr_{11}}{\sqrt{b^2 - s^2}} \bar{\alpha} + b\bar{r}_{10}, \quad s_0 = \bar{s}_0 = b\bar{s}_{10}, \quad (13)$$

and

$$r_{00} = \frac{s^2 \bar{\alpha}^2}{b^2 - s^2} r_{11} + 2 \frac{s \bar{\alpha}}{\sqrt{b^2 - s^2}} \bar{r}_{10} + \bar{r}_{00}, \quad (14)$$

$$\tilde{\theta} = t_1 \frac{s}{\sqrt{b^2 - s^2}} \bar{\alpha} - \frac{f'(b)}{f(b)} \bar{r}_{10} - \frac{f'(b)}{f(b)} \bar{s}_{10} - \eta. \quad (15)$$

Substituting (13), (14) and (15) into (8) and by using (11), we find that (8) is equivalent to the following equations:

$$\frac{\Phi}{2\Delta^2} (b^2 - s^2) \bar{r}_{00} = -\left\{ s \left(\frac{s\Phi}{2\Delta^2} - 2\Psi b^2 \right) r_{11} + (n+1)cb^2\phi - sbt_1 \right\} \bar{\alpha}^2, \quad (16)$$

$$\left(\frac{s\Phi}{\Delta^2} - 2\Psi b^2 \right) (r_{1A} + s_{1A}) - (b^2 Q + s) \frac{\Phi}{\Delta^2} s_{1A} + b\eta_A - bt_A = 0. \quad (17)$$

$$\eta_1 = 0. \quad (18)$$

Let

$$\Upsilon := \left[\frac{s\Phi}{\Delta^2} - 2\Psi b^2 \right]'$$

We see that $\Upsilon = 0$ if and only if

$$\frac{s\Phi}{\Delta^2} - 2\Psi b^2 = b^2\mu,$$

where $\mu = \mu(x)$ is independent of s .

Let us suppose that $\Xi = \frac{(b^2Q+s)\Phi}{\Delta^2}$ is not constant. Now we shall divide the proof into two cases: (i) $\Upsilon = 0$ and (ii) $\Upsilon \neq 0$.

3.1 $\Upsilon = 0$

First, note that $\Upsilon = 0$ implies that

$$\frac{s\Phi}{\Delta^2} - 2\Psi b^2 = b^2\mu, \quad (19)$$

where $\mu = \mu(x)$ is a function on M independent of s . First, we prove the following.

Lemma 3.2 *Let (M, F) be a n -dimensional Finsler manifold. Suppose that $F = \alpha\phi(\beta/\alpha)$ be an (α, β) -metric and $\Upsilon = 0$. If F has weak isotropic S -curvature, $\mathbf{S} = (n+1)cF + \eta$, then β satisfies*

$$r_{ij} = ka_{ij} - \varepsilon b_i b_j + \frac{1}{b^2}(r_i b_j + r_j b_i), \quad (20)$$

where $k = k(x)$, $\varepsilon = \varepsilon(x)$, and $\phi = \phi(s)$ satisfies the following ODE:

$$(k - \varepsilon s^2) \frac{\Phi}{2\Delta^2} = \left\{ \nu + (k - \varepsilon b^2)\mu \right\} s - (n+1)c\phi, \quad (21)$$

where $\nu = \nu(x)$. If $s_0 \neq 0$, then ϕ satisfies the following additional ODE:

$$\frac{\Phi}{\Delta^2}(Qb^2 + s) = b^2(\mu + \lambda), \quad (22)$$

where $\lambda = \lambda(x)$.

Proof: Since $\Phi \neq 0$ and \bar{r}_{00} and $\bar{\alpha}$ are independent of s , it follows from (16) and (17) that in a special coordinate system (s, y^a) at a point x , the following relations hold

$$r_{AB} = k\delta_{AB}, \quad (23)$$

$$s \left(\frac{s\Phi}{2\Delta^2} - 2\Psi b^2 \right) r_{11} + (n+1)cb^2\phi + k \frac{\Phi}{2\Delta^2}(b^2 - s^2) = bst_1, \quad (24)$$

$$\left(\frac{s\Phi}{\Delta^2} - 2\Psi b^2 \right) (r_{1A} + s_{1A}) - (b^2Q + s) \frac{\Phi}{\Delta^2} s_{1A} - bt_A = -b\eta_A, \quad (25)$$

where $k = k(x)$ is independent of s . Let

$$r_{11} = -(k - \epsilon b^2).$$

Then (20) holds. By (19), we have

$$\frac{s\Phi}{2\Delta^2} - 2\Psi b^2 = b^2\mu - \frac{s\Phi}{2\Delta^2}.$$

Then (24) and (25) become

$$b(k - \epsilon s^2)\frac{\Phi}{2\Delta^2} = st_1 + sb\mu(k - b^2\epsilon) - (n+1)cb\phi. \quad (26)$$

$$b^2\mu(r_{1A} + s_{1A}) - \frac{\Phi}{\Delta^2}(Qb^2 + s)s_{1A} - bt_A = -b\eta_A. \quad (27)$$

Letting $t_1 = b\nu$ in (26) we get (21). Now, suppose that $s_0 \neq 0$. Rewrite (27) as

$$\left\{b^2\mu - \frac{\Phi}{\Delta^2}(Qb^2 + s)\right\}s_{1A} = bt_A - b\eta_A - b^2\mu r_{1A}.$$

We can see that there is a function $\lambda = \lambda(x)$ on M such that

$$\mu b^2 - \frac{\Phi}{\Delta^2}(Qb^2 + s) = -b^2\lambda.$$

This gives (22).

Q.E.D.

Lemma 3.3 ([4]) *Let $F = \alpha\phi(\beta/\alpha)$ be an (α, β) -metric. Assume that*

$$\phi \neq k_1\sqrt{1 + k_2s^2} + k_3s$$

for any constants $k_1 > 0, k_2$ and k_3 . If $\Upsilon = 0$, then $b = \text{constant}$.

An (α, β) -metric is called Randers-type if $\phi = k_1\sqrt{1 + k_2s^2} + k_3s$ for any constants $k_1 > 0, k_2$ and k_3 . Now, we consider the equivalency of the notions weak isotropic S-curvature and isotropic S-curvature for a non-Randers type (α, β) -metric.

Lemma 3.4 *Let $F = \alpha\phi(\beta/\alpha)$ be a non-Randers type (α, β) -metric. Suppose that Ξ is not constant and $\Upsilon = 0$. Then F is of weak isotropic S-curvature if and only if F is of isotropic S-curvature.*

Proof: It is sufficient to prove that if F is of weak isotropic S-curvature, then F is of isotropic S-curvature. By $db = (r_0 + s_0)/b$ and Lemma 3.3, we have

$$r_0 + s_0 = 0.$$

Then by the formula of S-curvature of an (α, β) -curvature, we get

$$\mathbf{S} = -\alpha^{-1}\frac{\Phi}{2\Delta^2}\left\{r_{00} - 2\alpha Qs_0\right\}.$$

By Lemma 3.2,

$$r_{00} = (k - \varepsilon s^2)\alpha^2 + \frac{2s}{b^2}r_0\alpha.$$

Then

$$\mathbf{S} = -(k - \varepsilon s^2)\frac{\Phi}{2\Delta^2}\alpha + \frac{\Phi}{b^2\Delta^2}(b^2Q + s)s_0.$$

By (21), we have

$$\mathbf{S} = -s\left\{\nu + (k - \varepsilon b^2)\mu\right\}\alpha + \frac{\Phi}{b^2\Delta^2}(b^2Q + s)s_0 + (n+1)c\phi\alpha. \quad (28)$$

Since $\mathbf{S} = (n+1)cF + \eta$, then by (28) we obtain the following

$$-s\left\{\nu + (k - \varepsilon b^2)\mu\right\}\alpha + \frac{\Phi}{b^2\Delta^2}(b^2Q + s)s_0 = \eta. \quad (29)$$

Letting $y^i = \delta b^i$ for a sufficiently small $\delta > 0$ yields

$$-\delta\left\{\nu + (k - \varepsilon b^2)\mu\right\}b^2 = \delta\eta_i b^i.$$

It is easy to see that in the special coordinate $\eta_i b^i = 0$, hence in general $\eta_i b^i = 0$. We conclude that

$$\nu + (k - \varepsilon b^2)\mu = 0. \quad (30)$$

Then (29) reduce to

$$\frac{\Xi}{b^2}s_0 = \eta. \quad (31)$$

If $s_0 \neq 0$, then from the last equation, we obtain that Ξ is constant, which is excluded here. Hence, we have $s_0 = 0$. Thus by (31), we conclude that $\eta = 0$ and F has isotropic S-curvature $\mathbf{S} = (n+1)cF$. Q.E.D.

3.2 $\Upsilon \neq 0$

Here, we consider the case when $\phi = \phi(s)$ satisfies

$$\Upsilon \neq 0 \quad (32)$$

First we need the following

Lemma 3.5 *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an (α, β) -metric on an n -dimensional manifold. Assume that $\Upsilon \neq 0$. Suppose that F has weak isotropic S-curvature, $\mathbf{S} = (n+1)cF + \eta$. Then*

$$r_{ij} = ka_{ij} - \varepsilon b_i b_j - \lambda(s_i b_j + s_j b_i), \quad (33)$$

where $\lambda = \lambda(x)$, $k = k(x)$ and $\varepsilon = \varepsilon(x)$ are scalar functions of x and

$$-2s(k - \varepsilon b^2)\Psi + (k - \varepsilon s^2)\frac{\Phi}{2\Delta^2} + (n+1)c\phi - s\nu = 0, \quad (34)$$

where

$$\nu := -\frac{f'(b)}{bf(b)}(k - \varepsilon b^2). \quad (35)$$

If in addition $s_0 \neq 0$, i.e., $s_{A_o} \neq 0$ for some A_o , then

$$-2\Psi - \frac{Q\Phi}{\Delta^2} - \lambda\left(\frac{s\Phi}{\Delta^2} - 2\Psi b^2\right) = \delta, \quad (36)$$

where

$$\delta := -\frac{f'(b)}{bf(b)}(1 - \lambda b^2) - \frac{\eta_{A_o}}{s_{A_o}}. \quad (37)$$

Proof: By assumption, $\Phi \neq 0$. Similar to the proof of Lemma 3.2, it follows from (16) that there is a function $k = k(x)$ independent of s , such that

$$\bar{r}_{00} = k\bar{\alpha}^2, \quad (38)$$

$$s\left(\frac{s\Phi}{2\Delta^2} - 2\Psi b^2\right)r_{11} + (n+1)cb^2\phi + k\frac{\Phi}{2\Delta^2}(b^2 - s^2) = sbt_1. \quad (39)$$

Let

$$r_{11} = k - \varepsilon b^2,$$

where $\varepsilon = \varepsilon(x)$ is independent of s . By (12),

$$t_1 = b\nu,$$

where ν is given by (35). Plugging them into (39) yields (34).

Suppose that $s_0 = 0$. Then

$$bs_{1A} = s_A = 0.$$

Then (17) is reduced to

$$\left(\frac{s\Phi}{\Delta^2} - 2\Psi b^2\right)r_{1A} - bt_A = -b\eta_A. \quad (40)$$

By assumption, $\Upsilon \neq 0$, we know that $\frac{s\Phi}{\Delta^2} - 2\Psi b^2$ is not independent of s . It follows from (40) that

$$r_{1A} = 0, \quad t_A = \eta_A.$$

The above identities together with $r_{11} = k - \varepsilon b^2$ and $t_1 = b\nu$ imply the following identities

$$r_{ij} = ka_{ij} - \varepsilon b_i b_j. \quad (41)$$

Now, suppose that $s_0 \neq 0$. Then

$$s_{A_o} = bs_{1A_o} \neq 0$$

for some A_o . Differentiating (17) with respect to s yields

$$\Upsilon r_{1A} - \left[\frac{Q\Phi}{\Delta^2} + 2\Psi\right]' b^2 s_{1A} = 0. \quad (42)$$

Let

$$\lambda := -\frac{r_{1A_o}}{b^2 s_{1A_o}}.$$

Plugging it into (42) yields

$$-\lambda \Upsilon - \left[\frac{Q\Phi}{\Delta^2} + 2\Psi \right]' = 0. \quad (43)$$

It follows from (43) that

$$\delta := -\frac{Q\Phi}{\Delta^2} - 2\Psi - \lambda \left[\frac{s\Phi}{\Delta^2} - 2\Psi b^2 \right]$$

is a number independent of s . By assumption that $\Upsilon \neq 0$, we obtain from (42) and (43) that

$$r_{1A} + \lambda b^2 s_{1A} = 0. \quad (44)$$

(38) and (44) together with $r_{11} = k - \varepsilon b^2$ imply that

$$r_{ij} + \lambda(b_i s_j + b_j s_i) = k a_{ij} - \varepsilon b_i b_j. \quad (45)$$

By (12) and (44),

$$t_A = \frac{f'(b)}{f(b)}(b^2 \lambda - 1)s_{1A}.$$

On the other hand, by (17) and (44), we obtain

$$bt_A = \delta b^2 s_{1A} + b\eta_A.$$

Combining the above identities, we get (37).

Q.E.D.

Lemma 3.6 *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an (α, β) -metric. Suppose that $\phi = \phi(s)$ satisfies (32) and $\phi \neq k_1\sqrt{1+k_2s^2} + k_3s$ for any constants $k_1 > 0$, k_2 and k_3 . If F has weak isotropic S -curvature, then*

$$r_j + s_j = 0.$$

Proof: Suppose that $r_j + s_j \neq 0$, then $db = (r_0 + s_0)/b \neq 0$ and hence $b := \|\beta_x\|_\alpha \neq \text{constant}$ in a neighborhood. We view b as a variable in (34) and (36). Since $\phi = \phi(s)$ is a function independent of x , (34) and (36) actually give rise infinitely many ODEs on ϕ . First, we consider (34). Let

$$eq := \Delta^2 \left\{ -2s(k - \varepsilon b^2)\Psi + (k - \varepsilon s^2)\frac{\Phi}{2\Delta^2} + (n+1)c\phi - s\nu \right\}.$$

We have

$$eq = \Xi_0 + \Xi_2 b^2 + \Xi_4 b^4,$$

where Ξ_0, Ξ_2, Ξ_4 are independent of b and

$$\Xi_4 := \left\{ (\varepsilon - \nu)s + (n+1)c\phi \right\} \frac{\phi^2}{(\phi - s\phi')^4} (\phi'')^2.$$

It follows from (34) that $eq = 0$. Thus

$$\Xi_0 = 0, \quad \Xi_2 = 0, \quad \Xi_4 = 0.$$

Since $\phi'' \neq 0$, the equation $\Xi_4 = 0$ is equivalent to the following ODE

$$(\varepsilon - \nu)s + (n + 1)c\phi = 0.$$

We conclude that

$$\varepsilon = \nu, \quad c = 0.$$

Then by a direct computation we get

$$\Xi_0 + \Xi_2 s^2 = -\frac{1}{2}(1 + sQ)\left\{(n - 1)(k - \varepsilon s^2)(Q - sQ') + 2kQ + 2\varepsilon s\right\}.$$

By $\Xi_0 = 0$ and $\Xi_2 = 0$ it follows that

$$(n - 1)(k - \varepsilon s^2)(Q - sQ') + 2kQ + 2\varepsilon s = 0, \quad (46)$$

Suppose that $(k, \varepsilon) \neq 0$. We claim that $k \neq 0$. If this is not true, i.e., $k = 0$, then $\varepsilon \neq 0$ and (46) is reduced to

$$-(n - 1)s(Q - sQ') + 2 = 0.$$

Letting $s = 0$, we get a contradiction.

Now we have $k \neq 0$. It is easy to see that $Q(0) = 0$. Let

$$\tilde{Q} := Q(s) - sQ'(0).$$

Plugging it into (46) yields

$$(n - 1)(k - \varepsilon s^2)(\tilde{Q} - s\tilde{Q}') + 2k\tilde{Q} + 2(kQ'(0) + \varepsilon)s = 0. \quad (47)$$

Differentiating of the equation (47) with respect to s and then letting $s = 0$, yields

$$kQ'(0) + \varepsilon = 0.$$

Then (47) is reduced to following

$$(n - 1)(k - \varepsilon s^2)(\tilde{Q} - s\tilde{Q}') + 2k\tilde{Q} = 0.$$

Solving above ODE, we obtain

$$\tilde{Q} = c_1 s e^{\frac{2k}{n-1} \int_0^s \frac{1}{u(k - \varepsilon u^2)} du}.$$

Since $\tilde{Q}'(0) = 0$, we have $c_1 = 0$. Hence, $\tilde{Q} = 0$. We get

$$Q(s) - sQ'(0) = 0.$$

Then it follows that

$$Q(s) = Q'(0)s.$$

In this case, $\phi = c_1\sqrt{1+c_2s^2}$ where $c_1 > 0$ and c_2 are numbers independent of s . This case is excluded in the assumption. Therefore $k = 0$ and $\varepsilon = 0$. Then (33) is reduced to

$$r_{ij} = -\lambda(s_j b_i + s_i b_j).$$

Then

$$r_j + s_j = (1 - \lambda b^2)s_j.$$

At the beginning of the proof, we suppose that $r_j + s_j \neq 0$. Thus, we conclude that

$$1 - \lambda b^2 \neq 0, \quad \text{and} \quad s_j \neq 0.$$

By Lemma 3.5, $\phi = \phi(s)$ satisfies (36). Let

$$EQ := \Delta^2 \left\{ -2\Psi - \frac{Q\Phi}{\Delta^2} - \lambda \left(\frac{s\Phi}{\Delta^2} - 2\Psi b^2 \right) - \delta \right\}.$$

We have

$$EQ = \Omega_0 + \Omega_2 b^2 + \Omega_4 b^4,$$

where $\Omega_0, \Omega_2, \Omega_4$ are independent of b and

$$\Omega_4 = (Q')^2(\lambda - \delta).$$

By (36), $EQ = 0$. Thus

$$\Omega_0 = 0, \quad \Omega_2 = 0, \quad \Omega_4 = 0.$$

Since $Q' \neq 0$, $\Omega_4 = 0$ implies that

$$\delta = \lambda.$$

By a direct computation, we get

$$\Omega_0 + \Omega_2 s^2 = (1 + sQ) \left\{ (n+1)Q(Q - sQ') - Q' + \lambda \left[ns(Q - sQ') - 1 \right] \right\}.$$

The equations $\Omega_0 = 0$ and $\Omega_2 = 0$ imply that

$$\Omega_0 + \Omega_2 s^2 = 0,$$

that is

$$(n+1)Q(Q - sQ') - Q' + \lambda \left[ns(Q - sQ') - 1 \right] = 0.$$

We obtain

$$Q = -\frac{[k_0 n(n+1) - 1]\lambda s \pm \sqrt{\lambda k_0(k_0(1+n)^2 - 1 + \lambda s^2)}}{k_0(n+1)^2 - 1}.$$

Plugging it into $\Omega_2 = 0$ yields

$$k_0 \lambda = 0.$$

Then

$$Q = \frac{\lambda s}{k_0(n+1)^2 - 1}.$$

This implies that

$$\phi = k_1 \sqrt{1 + k_2 s^2},$$

where $k_1 > 0$ and k_2 are numbers independent of s . This case is excluded in the assumption of the lemma. Therefore, $r_j + s_j = 0$. Q.E.D.

Proposition 3.7 *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an (α, β) -metric. Suppose that $\phi = \phi(s)$ satisfies (32) and $\phi \neq k_1 \sqrt{1 + k_2 s^2} + k_3 s$ for any constants $k_1 > 0$, k_2 and k_3 . Suppose that Ξ is not constant. If F is of weak isotropic S -curvature, $\mathbf{S} = (n+1)cF + \eta$, then*

$$r_{ij} = \varepsilon(b^2 a_{ij} - b_i b_j), \quad s_j = 0, \quad (48)$$

where $\varepsilon = \varepsilon(x)$ is a scalar function on M and $\phi = \phi(s)$ satisfies

$$\varepsilon(b^2 - s^2) \frac{\Phi}{2\Delta^2} = -(n+1)c\phi. \quad (49)$$

Proof: Contracting (33) with b^i yields

$$r_j + s_j = (k - \varepsilon b^2) b_j + (1 - \lambda b^2) s_j. \quad (50)$$

By Lemma 3.6, $r_j + s_j = 0$. It follows from (50) that

$$(1 - \lambda b^2) s_j + (k - \varepsilon b^2) b_j = 0. \quad (51)$$

Contracting (51) with b^j yields

$$(k - \varepsilon b^2) b^2 = 0.$$

We get

$$k = \varepsilon b^2.$$

Then (33) is reduced to

$$r_{ij} = \varepsilon(b^2 a_{ij} - b_i b_j) - \lambda(b_i s_j + b_j s_i).$$

By (35),

$$\nu = 0.$$

Then (34) is reduced to (49).

We claim that $s_0 = 0$. Suppose that $s_0 \neq 0$. By (51), we conclude that

$$\lambda = \frac{1}{b^2}.$$

By (37),

$$\delta = -\frac{\eta_{A_o}}{s_{A_o}}.$$

It follows from (36) that

$$\frac{(b^2Q + s)\Phi}{\Delta^2} = \frac{b\eta_{A_o}}{s_{A_o}},$$

which implies that Ξ is constant. This is impossible by the assumption on non-constancy of Ξ . Therefore, $s_j = 0$. This completes the proof. Q.E.D.

By Proposition 3.7 and Lemma 2.3, we have the following.

Corollary 3.8 *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be a non-Randers type (α, β) -metric. Suppose that $\Upsilon \neq 0$ and Ξ is not constant. Then F is of weak isotropic S -curvature, if and only if it is of isotropic S -curvature.*

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